RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. THIRD SEMESTER EXAMINATION, DECEMBER 2019

SECOND YEAR [BATCH 2018-21]

Date : 11/12/2019 **MATHEMATICS [Honours]**

Time : 11 am – 3 pm **Paper : III** Full Marks : 100

[Use a separate Answer Book for each Group]

Group – A

Answer any five questions from Question Nos. 1 to 8 : [5×7]

1. a) Obtain a non-singular transformation that will reduce the real quadratic form $x^2+y^2+z^2-2xy-2yz-2zx$, into a normal form. [5]

   b) Give an example of a linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ such that $N(T)=R(T)$. [2]

2. For what value of K the planes $x+y+z=2$, $3x+y-2z=K$, $2x+4y+7z=K+2$ intersect in a line? Also find the equation of the line in that case. [5+2]

3. Solve the systems $AX=E_1$, $AX=E_2$, $AX=E_3$, where $A=\begin{pmatrix}1&1&1\\8&3&4\\2&1&1\end{pmatrix}, E_1=\begin{pmatrix}1\\0\\0\end{pmatrix}, E_2=\begin{pmatrix}0\\1\\0\end{pmatrix}, E_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$. Hence find A^{-1}. [5+2]

4. A linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ is defined by $T(0,1,1)=(1,0,1)$, $T(1,0,1)=(2,3,4), T(1,1,0)=(1,2,3)$. Find the maxtrix of T relative to the ordered basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbb{R}^3$. Deduce that T is invertible. Find the matrix of T^{-1} relative to the ordered basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbb{R}^3$. [3+1+3]

5. a) Let A,B be two matrices over the same field F such that AB is defined. Prove that rank of $AB\le\min\{rank\ of\ A, rank\ of\ B\}$. [4]

   b) Find a linear operation T on $\mathbb{R}^3$ such that

   $Ker\,T=\{(x,y,z)\in\mathbb{R}^3 : x+y+z=0\}$ [3]

6. Let θ be a real number. Prove that the following two matrices are similar over the field of complex number : $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}, \begin{bmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{bmatrix}$. [7]

7. a) Prove that two finite dimensional vector spaces V and W over a field F are isomorphic if and only if dim V = dim W. [5]

b) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that T is the reflection about the line $y=-x$. Find the matrix of T relative to the ordered basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$. [2]

8. a) Let W be a subspace of a vector space V over a field F and $\alpha,\beta \in V$. Prove that the cosets $\alpha+W=\beta+W$ if and only if $\alpha-\beta \in W$. [4]

b) Let V be a finite dimensional vector space and $T:V \to V$ be a linear transformation. If $V=R(T)+N(T)$, prove that $V=R(T)\oplus N(T)$ [3]

Answer <u>any three</u> questions from <u>Question no. 9 to 13</u> : [3 × 5]

9. Find the radius of convergence and the sum of the following power series:

a) $\sum_{n=1}^{\infty} nx^{n-1}$

b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ [2.5+2.5]

10. For each $n \in \mathbb{N}$, let $f_n(x)=x-\frac{1}{n}, g_n(x)=x+\frac{2}{n}$ on $[0,\infty)$. Show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ are uniformly convergent on $[0,\infty)$ but the sequence $\{f_n g_n\}_{n\in\mathbb{N}}$ is not so. [5]

11. A sequence $\{f_n, n\geq 1\}$ of real-valued functions defined on a non-empty set $T(\subset\mathbb{R})$ is said to be uniformly bounded on T if there exists a constant M > 0 such that $\left|f_n(x)\right| \leq M$ for all $x\in T$ and $n\geq 1$.

Suppose $\{f_n, n\geq 1\}$ is a sequence of real-valued function on a non-empty set $T(\subset\mathbb{R})$. Suppose each f_n is bounded on T and the sequence $\{f_n, n\geq 1\}$ converges uniformly on T to a function f. Show that the sequence is uniformly bounded on T. [5]

12. Assume that the series $\sum_{n=0}^{\infty} a_n$ is convergent and set $A_n=\sum_{i=0}^{n} a_i$. Prove that for $|x|<1$ the series $\sum_{n=0}^{\infty} A_n x^n$ converges and $\sum_{n=0}^{\infty} a_n x^n = (1-x)\sum_{n=0}^{\infty} A_n x^n$. [5]

13. a) State Weierstrass' approximation theorem. [1]

b) Find $\lim_{x\to 0}\sum_{n=1}^{\infty}\frac{\cos nx}{n(n+1)}$. [4]

Group – B

Answer any four questions from Question Nos. 14 to 19 : [4×5]

14. A variable plane has intercepts on the coordinate axes the sum of whose squares is K^2. Show that the locus of the foot of perpendicular from the origin to the plane is

$$\left(x^2+y^2+z^2\right)^2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=K^2.$$ [5]

15. A variable line always intersects the lines x = a, y = 0 ; y = a, z = 0 and z = a , x = 0. Prove that the equation of its locus is $xy+yz+zx-a(x+y+z-a)=0$. [5]

16. A sphere touches the planes $2x+3y-6z+14=0$ and $2x+3y-6z+42=0$ and its centre lies on the line $2x+z=0, y=0$. Find the equation of the sphere. [5]

17. The base (guiding curve) of a cone is $y^2=4ax, z=0,$ the vertex being at (0,0,2a). Prove that the cone is its own reciprocal. [5]

18. Find the equations of the generators of the hyperboloid $\frac{x^2}{25}+\frac{y^2}{16}-\frac{z^2}{4}=1$ which are parallel to the plane $8x+10y+20z-11=0$. [5]

19. Reduce the equation $4x^2-y^2-z^2+2yz-8x-4y+8z-2=0$ to its canonical form and hence classify the conicoid represented by it. [4+1]

Answer any two questions from Questions nos. 20 to 22: [2×12]

20. a) A particle is projected from an apse at a distance a with a velocity from infinity under the action of a central acceleration $\frac{\mu}{r^{2n+3}}, \mu>0$. Prove that the path is $r^n=a^n\cos n\theta$. [6]

b) If the earth's attraction vary inversely as the square of the distance from the centre and g be its magnitude at the surface , show that the time of falling of a particle from a height h above the surface to the surface is $\sqrt{\frac{a+h}{2g}}\left[\frac{a+h}{a}\cos^{-1}\sqrt{\frac{a}{a+h}}+\sqrt{\frac{h}{a}}\right]$, where a is the radius of the earth. [6]

21. a) Find the tangential and normal components of velocity and acceleration of a particle which describes a plane curve. [7]

b) A particle rests in equilibrium under the attractions of two centres of force which attract directly as the distance, their attractions per unit of mass at unit distance being μ_1 and μ_2. The particle is slightly displaced towards one of them. Prove that the motion is oscillatory of period $\frac{2\pi}{\sqrt{\mu_1+\mu_2}}$. [5]

22. a) A particle is projected with velocity u at an inclination α (acute) above the horizon in a medium that resists the motion by a force kv per unit mass, where v is the velocity of the particle. Find the equation of the path of the particle. [6]

b) A particle is placed at rest in a rough tube at a distance *a* from one end and the tube starts rotating horizontally with a uniform angular velocity ω about this end. Show that the distance of the particle at time t is $ae^{-\omega t\tan\varepsilon}\left[\cosh(\omega t\sec\varepsilon)+\sin\varepsilon.\sinh(\omega t\sec\varepsilon)\right]$,where $\tan\varepsilon$ is the coefficient of friction. [6]

Answer <u>any one</u> question from question <u>nos. 23 to 24</u> : [1×6]

23. One end of an elastic string of unstretched length *a* is tied to a point on the top of a smooth table, and a particle attached to the other end move freely on the table. If the path be nearly circular of radius b, show that its apsidal angle is approximately $\pi\sqrt{\frac{b-a}{4b-3a}}$. [6]

24. A particle falls from rest under gravity in a medium whose resistance is k(velocity)^2. Show that at any time t (i) $v = c\tanh\left(\frac{gt}{c}\right)$ and (ii) $x=\left(\frac{c^2}{g}\right)\log\cosh\left(\frac{gt}{c}\right)$, where c is the terminal velocity, v is the velocity of the particle and x the distance descended. [6]

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